The Logistic Model --- Statement, Solution, and Application

A population growth function **P** with **Growth Rate k,** with **Carrying Capacity M,** and with **Initial Population P(0) = $\mathbf{P_0}$,** satisfies **The Logistic Model** if it is a solution of the

Logistic Differential Equation: $$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right) .$$

The formula for this solution P(t) is:

$$P(t) = \frac{M}{1 + Ae^{-kt}} \quad \text{where} \quad A = \frac{M - P_0}{P_0} .$$

An Application using the solution of the Logistic Differential Equation:

A population obeying the logistic equation begins with 1,000 bacteria, and then it doubles itself in 10 hours.

The population is observed eventually to stabilize at 20,000 bacteria.

(A) Find the number of bacteria in the population after 25 hours and

(B) find the time it takes to reach 1/2 of the carrying capacity.

Solution: Let k = the population's growth rate and M = the carrying capacity.

Let $P(t)$ = the number of bacteria in the population after t hours.

From the information given, M = 20,000 and the initial population is

$P_0 = P(0) = 1{,}000.$ Also, $P(10) = 2{,}000.$

Recall: $P(t) = \dfrac{M}{1 + Ae^{-kt}}$ where $A = \dfrac{M - P_0}{P_0}$.

$P_0 = P(0) = 1{,}000$; $P(10) = 2{,}000$; $M = 20{,}000$

$$A = \frac{20{,}000 - 1{,}000}{1{,}000} = \frac{20-1}{1} = 19 \qquad \boxed{A = 19}$$

$$P = \frac{20{,}000}{1 + 19e^{-kt}}.$$

NEXT, WE FIND k.

$$P(10) = 2{,}000 = \frac{20{,}000}{1 + 19e^{-k(10)}}$$

$$(2{,}000)(1 + 19e^{-k(10)}) = 20{,}000 = 10(2{,}000)$$

$$1 + 19e^{-10k} = 10$$

$$19e^{-10k} = 9$$

$$e^{-10k} = 9/19 \Rightarrow -10k = \ln(9/19)$$

$$10k = -\ln(9/19) = \ln(19/9)$$

$$\boxed{k = \tfrac{1}{10}\ln(19/9)}$$

$$\boxed{P = \frac{20{,}000}{1 + 19e^{(-\frac{1}{10})\ln(19/9)t}}}$$

(A) For $t = 25$ hours,

$$P(25) = \frac{20{,}000}{1 + 19e^{(-2.5)\ln(19/9)}} = 5083.75$$

$\boxed{\text{After 25 hours, the population is 5,084 bacteria}}$

(B). Find the time it takes to reach ½ of the carrying capacity.

$(\frac{1}{2})(20{,}000) = 10{,}000$ and

$$\text{Recall: } P(t) = \frac{20{,}000}{1 + 19e^{(-\frac{1}{10})\ln(19/9)t}}$$

$$\text{Solve } \quad 10{,}000 = \frac{20{,}000}{1+19e^{(-\frac{1}{10})\ln(19/9)t}} \quad \text{for } t.$$

$$(10{,}000)(1 + e^{(-\frac{1}{10})\ln(19/9)t}) = 20{,}000$$

$$1 + 19e^{(-\frac{1}{10})\ln(19/9)t} = 2$$

$$19e^{(-\frac{1}{10})\ln(19/9)t} = 1$$

$$e^{(-\frac{1}{10})\ln(19/9)t} = 1/19 \qquad \text{TAKING LOGARITHMS, ...}$$

$$-\frac{1}{10}\ln(19/9)t = \ln(1/19)$$

$$\ln(19/9)t = -10\ln(1/19) = 10\ln(19)$$

$$t = \frac{10\ln(19)}{\ln(19/9)} = \frac{10\ln(19)}{\ln(19) - \ln(9)}$$

$t = 39.4055$ (Rounded to 4 decimal places)

$\boxed{\text{It takes 39.4055 hours for the population to reach ½ its carrying capacity.}}$